

# ENUMERATING $(\mathbf{2} + \mathbf{2})$ -FREE POSETS BY INDISTINGUISHABLE ELEMENTS

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**ABSTRACT.** A poset is said to be  $(\mathbf{2} + \mathbf{2})$ -free if it does not contain an induced subposet that is isomorphic to  $\mathbf{2} + \mathbf{2}$ , the union of two disjoint 2-element chains. Two elements in a poset are indistinguishable if they have the same strict up-set and the same strict down-set. Being indistinguishable defines an equivalence relation on the elements of the poset. We introduce the statistic *maxindist*, the maximum size of a set of indistinguishable elements. We show that, under a bijection of Bousquet-Mélou et al. [1], indistinguishable elements correspond to letters that belong to the same run in the so-called ascent sequence corresponding to the poset. We derive the generating function for the number of  $(\mathbf{2} + \mathbf{2})$ -free posets with respect to both *maxindist* and the number of different strict down-sets of elements in the poset. Moreover, we show that  $(\mathbf{2} + \mathbf{2})$ -free posets  $P$  with *maxindist*( $P$ ) at most  $k$  are in bijection with upper triangular matrices of nonnegative integers not exceeding  $k$ , where each row and each column contains a nonzero entry. (Here we consider isomorphic posets to be equal.) In particular,  $(\mathbf{2} + \mathbf{2})$ -free posets  $P$  on  $n$  elements with *maxindist*( $P$ ) = 1 correspond to upper triangular binary matrices where each row and column contains a nonzero entry, and whose entries sum to  $n$ . We derive a generating function counting such matrices, which confirms a conjecture of Jovovic [8], and we refine the generating function to count upper triangular matrices consisting of nonnegative integers not exceeding  $k$  and having a nonzero entry in each row and column. That refined generating function also enumerates  $(\mathbf{2} + \mathbf{2})$ -free posets according to *maxindist*. Finally, we link our enumerative results to certain restricted permutations and matrices.

## 1. INTRODUCTION

This paper continues the study of enumerative properties of three distinct equinumerous classes of combinatorial objects, namely,  $(\mathbf{2} + \mathbf{2})$ -free posets (also known as *interval orders*, see Fishburn [5]), ascent sequences, and upper triangular matrices with nonnegative integer entries and where each row and column contains a nonzero entry. We build on the work of Bousquet-Mélou et al. [1], who presented a bijection between  $(\mathbf{2} + \mathbf{2})$ -free posets and ascent sequences, and that of Dukes and Parviainen [3], who gave a bijection between ascent sequences and upper triangular matrices with nonnegative integer entries and no rows or columns of all zeros.

It is important to note that, as in [1], we consider, and count,  $(\mathbf{2} + \mathbf{2})$ -free posets up to isomorphism. That is, we consider two such posets to be equal if there is an order preserving bijection between them. In [1] the isomorphism classes are referred to as “unlabeled posets”.

The central result of this paper is the determination of the generating function for the number of ascent sequences of length  $n$  with  $k$  pairs of consecutive elements that are equal. We call an ascent sequence with no two consecutive equal entries a *primitive ascent sequence*. A special case gives the generating function for the number of primitive ascent sequences. We show that under the bijections

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mentioned above, primitive ascent sequences correspond to *primitive*  $(\mathbf{2} + \mathbf{2})$ -free posets, defined by having no pair of elements with the same strict down-sets and the same strict up-sets, and also to upper triangular binary matrices with no rows or columns of zeros. This allows us to prove a conjecture of Jovovic [8] which states that the generating function for the number of upper triangular binary matrices with no rows or columns of zeros is given by

$$K(t) = \sum_{n \geq 0} \prod_{i=1}^n \left( 1 - \frac{1}{(1+t)^i} \right). \quad (1)$$

In order to state our results more precisely, we now introduce the three main classes of combinatorial structures treated in the paper, namely, ascent sequences,  $(\mathbf{2} + \mathbf{2})$ -free posets, and upper triangular matrices with nonnegative integer entries and no rows or columns of all zeros.

**1.1. Ascent sequences.** An *ascent* in an integer sequence  $(x_1, \dots, x_i)$ , is a  $j$  such that  $x_j < x_{j+1}$ . The number of ascents in such a sequence  $X$  is denoted  $\text{asc}(X)$ .

A sequence  $(x_1, \dots, x_n) \in \mathbb{N}^n$  is an *ascent sequence of length  $n$*  if and only if it satisfies  $x_1 = 0$  and

$$0 \leq x_i \leq 1 + \text{asc}(x_1, \dots, x_{i-1})$$

whenever  $2 \leq i \leq n$ .

Let  $\text{Asc}_n$  be the collection of ascent sequences of length  $n$  and let  $\text{Asc}$  be the collection of all ascent sequences, including the empty ascent sequence. If  $a \in \text{Asc}_n$  then we will write  $|a| = n$ . For example,  $(0, 1, 0, 2, 3, 1, 0, 0, 2)$  is an ascent sequence in  $\text{Asc}_9$ .

A *run* in an ascent sequence is a maximal subsequence of consecutive letters that are all equal. Let  $\text{Asc}^{(k)}$  be the collection of ascent sequences whose runs have length at most  $k$ , and let  $\text{Asc}_n^{(k)}$  be those  $a \in \text{Asc}^{(k)}$  that have  $|a| = n$ . A *primitive ascent sequence* is an ascent sequence with no runs of length greater than 1. Thus,  $\text{Asc}_n^{(1)}$  is the set of all primitive ascent sequences.

Given  $a = (a_1, \dots, a_n) \in \text{Asc}_n$ , we call a pair  $(a_i, a_{i+1})$  with  $a_i = a_{i+1}$  an *equal pair* of the sequence<sup>1</sup>. We denote the number of equal pairs in a sequence  $a$  by  $\text{epairs}(a)$ . For example  $\text{epairs}(0, 0, 0, 0, 0, 1, 1, 2, 1, 1) = 6$  since  $(a_1, a_2) = (a_2, a_3) = (a_3, a_4) = (a_4, a_5) = (0, 0)$  and  $(a_6, a_7) = (a_9, a_{10}) = (1, 1)$ .

**1.2.  $(\mathbf{2} + \mathbf{2})$ -free posets.** Recall that we consider two posets to be equal if they are isomorphic. A poset is said to be  $(\mathbf{2} + \mathbf{2})$ -free if it does not contain an induced subposet that is isomorphic to  $\mathbf{2} + \mathbf{2}$ , the union of two disjoint 2-element chains. We let  $\mathbf{P}$  denote the set of  $(\mathbf{2} + \mathbf{2})$ -free posets (including the empty poset) and  $\mathbf{P}_n$  the set of all such posets on  $n$  elements. For  $P \in \mathbf{P}$ , let  $|P|$  be the number of elements in  $P$ .

An important characterization (see [5, 6, 11]) says that a poset is  $(\mathbf{2} + \mathbf{2})$ -free if and only if its strict down-sets can be ordered linearly by inclusion. For a poset  $P = (P, \prec_P)$  and  $x \in P$ , the strict down-set of  $x$ , denoted  $D(x)$ , is the set of all  $y \in P$  such that  $y \prec_P x$ . Clearly, any poset is uniquely specified by listing the collection of strict down-sets of each element. The *trivial down-set* is the empty set. Thus if  $P$  is a  $(\mathbf{2} + \mathbf{2})$ -free poset, we can write  $D(P) = \{D(x) : x \in P\}$  as

$$D(P) = (D_0, D_1, \dots, D_k)$$

where  $\emptyset = D_0 \subset D_1 \subset \dots \subset D_k$ . We then say that  $x \in P$  is at *level  $i$*  if  $D(x) = D_i = D_i(P)$  and write  $\ell(x) = i$ . We also define  $\text{levels}(P)$  by setting  $\text{levels}(P) = k$ , where  $k$  is the index of the highest level in  $P$ .

<sup>1</sup>This is sometimes called a *level* in the literature on sequences, not to be confused with the definition of level in the present paper.

We denote by  $L_i(P) = \{x \in P : \ell(x) = i\}$  the set of all elements at level  $i$  and we set

$$L(P) = (L_0(P), L_1(P), \dots, L_{\text{levels}(P)}(P)).$$

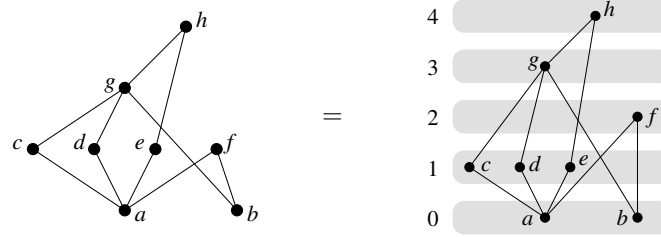
Let  $m_P$  be a maximal element of  $P$  whose strict down-set is the smallest of the strict down-sets of  $P$ 's elements. This element may not be unique but all such elements belong to the same level. We define  $\text{minmax}(P)$  by setting  $\text{minmax}(P) = \ell(m_P)$ . The maximal elements of  $P$  are  $P \setminus D_{\text{levels}(P)}(P)$ . Thus  $\text{minmax}(P) = \min\{\ell(x) : x \in P \setminus D_{\text{levels}(P)}(P)\}$ .

As a counterpart to the strict down-set  $D(x)$  of an element  $x$  in a poset, we let  $U(x)$  denote the *strict up-set* of  $x$ , that is,  $U(x) = \{y : x \prec_p y\}$ . Given  $P \in \mathbf{P}_n$ , we say that two elements  $x, y \in P$  are *indistinguishable* if  $D(x) = D(y)$  and  $U(x) = U(y)$ . We write this as  $x \sim_P y$  and note that  $\sim_P$  is an equivalence relation on  $P$ . Let us define  $\text{maxindist}(P)$  to be the size of the largest equivalence class in  $P$ .

For example, the elements  $c$  and  $d$  in the poset of Example 1 below, as well as the elements 2 and 3 in the poset of Example 3, are indistinguishable. We say that a  $(\mathbf{2} + \mathbf{2})$ -free poset is *primitive* if it contains no pair of indistinguishable elements.

We let  $\mathbf{P}_n^{(k)}$  denote the set of all  $(\mathbf{2} + \mathbf{2})$ -free posets  $P$  on  $n$  elements for which  $\text{maxindist}(P)$  is at most  $k$ . In particular,  $\mathbf{P}_n^{(1)}$  is the set of primitive  $(\mathbf{2} + \mathbf{2})$ -free posets on  $n$  elements. We define the statistic  $\text{rep}(P)$  for  $P \in \mathbf{P}$  to be the minimum number of elements that need to be removed to create a primitive poset. For example, the value of this statistic is 1 on the posets in both Examples 1 and 3. Note that  $\text{maxindist}(P) = 1$  if and only if  $\text{rep}(P) = 0$ .

**Example 1.** Let  $P$  be the following  $(\mathbf{2} + \mathbf{2})$ -free poset (reproduced from [1] with kind permission of the authors):



On the right the poset has been redrawn to show the level numbers determined by the strict down-set of each element (when compared to the strict down-sets of other elements). Notice that  $\text{levels}(P) = 4$ . The strict down-set, level, and strict up-set of each element is as follows:

$x$	$a$	$b$	$c$	$d$	$e$	$f$	$g$	$h$
$D(x)$	$\emptyset$	$\emptyset$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b, c, d\}$	$\{a, b, c, d, e, g\}$
$\ell(x)$	0	0	1	1	1	2	3	4
$U(x)$	$\{c, d, e, f, g, h\}$	$\{f, g, h\}$	$\{g, h\}$	$\{g, h\}$	$\{h\}$	$\emptyset$	$\{h\}$	$\emptyset$

We therefore have  $D(a) = D(b) \subset D(c) = D(d) = D(e) \subset D(f) \subset D(g) \subset D(h)$ . The strict down-sets for each level are listed along with the elements of each level:

$i$	0	1	2	3	4
$D_i(P)$	$\emptyset$	$\{a\}$	$\{a, b\}$	$\{a, b, c, d\}$	$\{a, b, c, d, e, g\}$
$L_i(P)$	$\{a, b\}$	$\{c, d, e\}$	$\{f\}$	$\{g\}$	$\{h\}$

The maximal elements of  $P$  are  $P \setminus D_4(P) = \{f, h\}$ . Since  $D(f) \subset D(h)$  we have  $m_P = f$  and  $\text{minmax}(P) = 2$ .

**1.3. Upper triangular matrices.** Let  $M_n$  be the set of upper triangular matrices of nonnegative integers such that no row or column contains all zero entries, and the sum of the entries is  $n$ . Let  $M$  be the set of all such matrices, that is,  $M = \bigcup_{n \geq 0} M_n$ . For example,  $M_3$  consists of the following 5 matrices:

$$(3), \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For any  $A \in M$ , we let  $|A|$  be the sum of the entries in  $A$ , and we set  $\text{extra}(A) = |A| - \text{NZ}(A)$ , where  $\text{NZ}(A)$  is the number of nonzero entries in  $A$ . Also, let  $\text{index}(A)$  be the smallest value of  $i$  such that  $A_{i, \dim(A)}$  is nonzero, where  $\dim(A)$  is the number of rows (or columns) in  $A$ . As an example, let

$$A = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Then  $|A| = 1 + 3 + 2 + 5 + 2 = 13$ ,  $\text{NZ}(A) = 5$ ,  $\text{extra}(A) = 13 - 5 = 8$ , and  $\text{index}(A) = 3$  since the topmost non-zero entry in the final column is in the third row. Let  $M_n^{(k)}$  be the collection of matrices in  $M_n$  that have no entries exceeding  $k$ . In particular,  $M_n^{(1)}$  is the set of binary matrices in  $M_n$ .

**1.4. Enumerative results.** Let  $p_n$  be the number of  $(\mathbf{2} + \mathbf{2})$ -free posets on  $n$  elements. Bousquet-Mélou et al. [1] showed that the generating function for the number  $p_n$  of  $(\mathbf{2} + \mathbf{2})$ -free posets on  $n$  elements is

$$P(t) = \sum_{n \geq 0} p_n t^n = \sum_{n \geq 0} \prod_{i=1}^n (1 - (1-t)^i). \quad (2)$$

A more general power series  $F(t, u, v)$  that takes into account the statistics *number of levels* and *level number of the lowest maximal element* is implied by inserting the power series given in [1, Proposition 15] into [1, Lemma 14]. See [1, Section 6] for an overview of these generating functions. More recently, Kitaev and Remmel [9] generalized the result of [1, Section 6] to derive a generating function that incorporated two further statistics related to  $(\mathbf{2} + \mathbf{2})$ -free posets.

**1.5. Statements of main results.** In this paper we study the generating function

$$G(u, v, y, t) = \sum_{P \in \mathcal{P}} u^{\text{levels}(P)} v^{\text{minmax}(P)} y^{\text{rep}(P)} t^{|P|}.$$

Using the bijections of Bousquet-Mélou et al. [1] and Dukes and Parviainen [3], respectively, this is also the generating function of several statistics on ascent sequences and matrices. (This is made clear at the beginning of Section 5.)

We show that  $H(u, v, y, t) = G(u, v, y, t) - 1$  satisfies the following recurrence:

$$H(u, v, y, t)(v-1-t-tyv+ty+tuv) = t(v-1)-tH(u, 1, y, t)+tuv^2H(uv, 1, y, t). \quad (3)$$

Using the kernel method, we then show that

$$G(u, 1, y, t) = 1 + \frac{t(1-u)}{\Delta_1} + \sum_{n=1}^{\infty} \frac{t(1-u)(1-ty)^n(1+t-ty)^n \prod_{i=1}^n \Gamma_i}{\Delta_n \Delta_{n+1}}, \quad (4)$$

where  $\Delta_k = (1-ty)^k(1-u) + u(1+t-ty)^k$  and  $\Gamma_k = (u(1+t-ty)^k)/\Delta_k$ . We can then use (3) and (4) to give an explicit formula for  $G(u, v, y, t)$ .

We also show that the generating function for primitive  $(\mathbf{2} + \mathbf{2})$ -free posets is given by

$$K(t) = \sum_{n \geq 0} \prod_{i=1}^n \left( 1 - \frac{1}{(1+t)^i} \right) \quad (5)$$

which confirms a conjecture of Jovovic [8]. Primitive  $(\mathbf{2} + \mathbf{2})$ -free posets are of special interest as one can easily generate all  $(\mathbf{2} + \mathbf{2})$ -free posets from the primitive ones by specifying the number of copies of each element.

Finally, we show that  $(\mathbf{2} + \mathbf{2})$ -free posets for which the maxindist statistic is at most  $k$  correspond to ascent sequences with runs of length at most  $k$ , and to upper-triangular matrices with entries not exceeding  $k$ . This allows us to generalize formula (5) to prove that

$$\sum_{n \geq 0} |\mathcal{P}_n^{(k)}| x^n = \sum_{n \geq 0} |\mathcal{M}_n^{(k)}| x^n = \sum_{n \geq 0} |\mathcal{Asc}_n^{(k)}| x^n = \sum_{n \geq 0} \prod_{i=1}^n \left( 1 - \left( \frac{1-x}{1-x^k} \right)^i \right).$$

**1.6. Outline of the paper.** In Section 2 we recall the bijection of Bousquet-Mélou et al. [1] between  $(\mathbf{2} + \mathbf{2})$ -free posets and ascent sequences. In Subsection 2.2 we show that  $|\mathcal{Asc}_n^{(k)}| = |\mathcal{P}_n^{(k)}| = |\mathcal{M}_n^{(k)}|$ . In Section 3 we derive the generating function for primitive ascent sequences and for ascent sequences with runs of length at most  $k$ . In Section 4 we derive our formula for  $G(u, v, y, t)$  and discuss its specialization  $G(u, 1, 0, t)$  corresponding to primitive ascent sequences. Finally, in Section 5, we show that restricting ascent sequences by bounding the run-length corresponds, via the bijection in [1], to bounding the length of a sequence of consecutive descents on the restricted permutations in [1, Sect. 2]. We also show that a similar statement holds for the relationship between ascent sequences and Stoimenow matchings.

## 2. $(\mathbf{2} + \mathbf{2})$ -FREE POSETS, ASCENT SEQUENCES AND MATRICES

**2.1. Constructing  $(\mathbf{2} + \mathbf{2})$ -free posets from ascent sequences.** In this subsection we review the essential parts of [1, Section 3] with proofs omitted. We describe a bijective map  $\mathfrak{B}$  from the collection of ascent sequences of length  $n$  to the collection of (canonically labeled)  $(\mathbf{2} + \mathbf{2})$ -free posets on  $n$  elements. The mapping is a step by step procedure which constructs a poset element by element. One always starts with the single poset having one element labeled ‘1’. The  $j$ th element of the poset to be inserted is labeled ‘ $j$ ’.

Central to the construction are the three addition rules: **Add1**, **Add2** and **Add3**. Given a poset  $P \in \mathcal{P}_m$ , and a value  $i \in [0, 1 + \text{levels}(P)]$ , we produce a poset  $\Phi(P, i) \in \mathcal{P}_{m+1}$  where the new poset element, regardless of its position, has label  $m+1$ . The appropriate addition rule to use depends on whether  $i \in [0, \text{minmax}(P)]$ ,  $i = 1 + \text{levels}(P)$  or  $i \in [\text{minmax}(P) + 1, \text{levels}(P)]$ .

Since a  $(\mathbf{2} + \mathbf{2})$ -free poset  $P$  is uniquely determined by the pair  $(D(P), L(P))$ , in defining the addition operations below it suffices to only specify how  $D(P)$  and  $L(P)$  change. Note that **Add1** leaves  $\text{levels}(P)$  unchanged, whereas **Add2** and **Add3** increase  $\text{levels}(P)$  by one.

Given  $P \in \mathcal{P}_n$ , let us write  $D_i = D_i(P)$  and  $L_i = L_i(P)$ . Given a value  $i$  with  $0 \leq i \leq 1 + \text{levels}(P)$ , let  $\Phi(P, i)$  be the poset  $Q$  obtained from  $P$  in the following way:

(Add1) If  $0 \leq i \leq \text{minmax}(P)$  then set  $D(Q) = D(P)$  and

$$L(Q) = (L_0, \dots, L_i \cup \{n+1\}, \dots, L_{\text{levels}(P)}).$$

(Add2) If  $i = 1 + \text{levels}(P)$  then set

$$\begin{aligned} D(Q) &= (D_0, \dots, D_{\text{levels}(P)}, P) \\ L(Q) &= (L_1, \dots, L_{\text{levels}(P)}, \{n+1\}). \end{aligned}$$

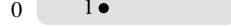
(Add3) If  $\text{minmax}(P) < i < 1 + \text{levels}(P)$  then set

$$\begin{aligned} \mathcal{M} &= L_0 \cup \dots \cup L_{i-1} \setminus D_{\text{levels}(P)} \\ D(Q) &= (D_0, \dots, D_i, D_i \cup \mathcal{M}, \dots, D_{\text{levels}(P)} \cup \mathcal{M}) \\ L(Q) &= (L_0, \dots, L_{i-1}, \{n+1\}, L_i, \dots, L_{\text{levels}(P)}). \end{aligned}$$

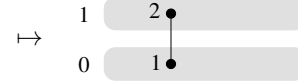
An important property of the above addition operations is that  $\text{minmax}(\Phi(P, i)) = i$ , since all maximal elements below level  $i$  are covered and therefore not maximal in  $\Phi(P, i)$ . Note that the single poset  $P \in P^{(1)}$  is such that  $D(P) = (\emptyset)$ ,  $L(P) = (\{1\})$  and  $\text{levels}(P) = 0$ .

**Definition 2.** Given  $x = (x_1, \dots, x_n) \in \text{Asc}_n$ , let  $\mathfrak{B}(x) = P^{(n)}$  where  $P^{(m)} := \Phi(P^{(m-1)}, x_m)$  for all  $1 < m \leq n$ .

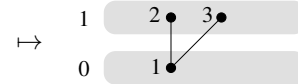
**Example 3.** In this example we construct the poset  $P = \mathfrak{B}(x)$  where  $x = (0, 1, 1, 0, 2, 0, 1) \in \text{Asc}_7$ . We begin from the poset  $P^{(1)}$  with just a single element, and successively construct  $P^{(2)}, \dots, P^{(7)} = P$  according to the addition rules. The poset  $P^{(1)}$  is the poset with one element labeled '1'. This element is the only element at level 0 of  $P^{(1)}$ , illustrated as follows:



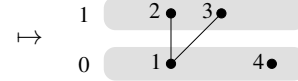
Since  $1 = 1 + \text{levels}(P^{(1)})$ , the poset  $P^{(2)} = \Phi(P^{(1)}, 1)$  is constructed by applying rule Add2. The new element is labeled '2':



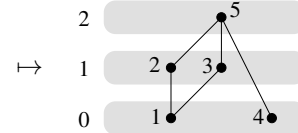
Since  $1 \in [0, \text{minmax}(P^{(2)})]$ , the poset  $P^{(3)} = \Phi(P^{(2)}, 1)$  is constructed by applying Add1:



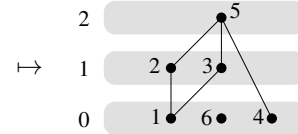
The poset  $P^{(4)} = \Phi(P^{(3)}, 0)$  is constructed by applying Add1:



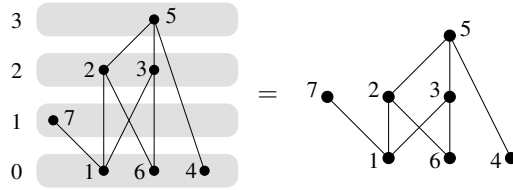
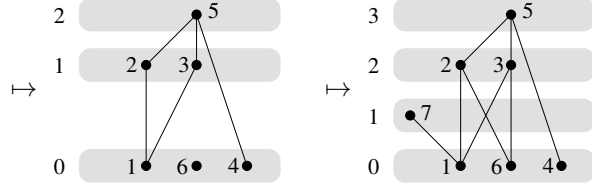
The poset  $P^{(5)} = \Phi(P^{(4)}, 2)$  is constructed by applying Add2:



The poset  $P^{(6)} = \Phi(P^{(5)}, 0)$  is constructed by applying Add1:



The poset  $P^{(7)} = \Phi(P^{(6)}, 1)$  is constructed by applying Add3. Note that we introduce a new empty level between levels  $x_6 - 1$  and  $x_6$  and insert a new single element with the same downset as the elements that were on that level. Then all the elements above it have the set  $\mathcal{M} = \{6\}$  included in their downsets.



Finally  $P = P^{(7)}$  and we have the canonically labeled poset  $\mathfrak{B}(x) \in \mathcal{P}_7$ .

**2.2. Bounded run lengths in ascent sequences.** In this subsection we prove Propositions 4 and 7 establishing the relation between runs in ascent sequences, indistinguishable elements in  $(2 + 2)$ -free posets, and entries of restricted upper-triangular matrices. To be more precise, we show that

$$|\text{Asc}_n^{(k)}| = |\mathcal{P}_n^{(k)}| = |\mathcal{M}_n^{(k)}|.$$

We use the following proposition to deal with ascent sequences in order to obtain results for posets.

**Proposition 4.** *Let  $x \in \text{Asc}_n$  and  $P \in \mathcal{P}_n$  with  $P = \mathfrak{B}(x)$ . Given  $i < j$  we have that  $i \sim_P j$  if and only if  $x_i = x_{i+1} = \dots = x_j$ .*

*Proof.* We first show that  $i \sim_P (i + 1)$  iff  $x_i = x_{i+1}$ . Let  $x_i = x_{i+1}$ . Think of  $P$  being created by adding elements one by one and using the rules (Add1)–(Add3) and assume that  $i + 1$  has just entered  $P$  ( $i$  has already been added to  $P$  on the previous step). Since  $x_i = x_{i+1}$ , at this point  $(D(i), U(i)) = (D(i + 1), U(i + 1))$  where  $U(i) = U(i + 1) = \emptyset$ . Moreover, from the definitions of (Add1)–(Add3),  $D(i)$  and  $D(i + 1)$  will either both stay unchanged or will be changing in the same way while adding extra elements to  $P$ . The same applies to  $U(i)$  and  $U(i + 1)$ . Thus,  $i \sim_P (i + 1)$ . On the other hand, if  $x_i \neq x_{i+1}$ , then  $D(i) \neq D(i + 1)$  after adding  $i + 1$  to  $P$  ( $i$  and  $(i + 1)$  will be on different levels) and the definitions of (Add1)–(Add3) guarantee that  $i$  and  $(i + 1)$  will remain on different levels while adding extra elements to  $P$ . That is,  $i \not\sim_P (i + 1)$ .

Next we show that if  $x_i = x_j$  and there exists  $x_k \neq x_i$  such that  $i < k < j$  then  $i \not\sim_P j$ . To prove this we need the notion of the *modified ascent sequence*  $\hat{x}$  and its properties introduced in [1, Section 4]. If  $x_i x_{i+1} \dots x_j$  contains an element  $x_s > x_i$  then we can take the minimum such  $s$  to see that  $s \in U(i)$  but  $s \notin U(j)$  showing that  $i \not\sim_P j$ . Otherwise, there must exist an ascent  $x_s x_{s+1}$  with  $x_{s+1} \leq x_i$  and  $i < s < j$ . This would mean that in  $\hat{x} = \hat{x}_1 \hat{x}_2 \dots \hat{x}_n$ , we have  $\hat{x}_i > \hat{x}_j$ , so  $i$  will be on a higher level than  $j$  in  $P$  and  $i \not\sim_P j$ .

To complete the proof we show that if  $i \not\sim_P j$  then either  $x_i \neq x_j$  or  $x_i = x_j$  but there exists  $x_k \neq x_i$  such that  $i < k < j$ . This, however, is a direct corollary to the definition and properties of the modified ascent sequence  $\hat{x}$  whose maximal runs of

equal elements correspond to the level distribution of elements in  $P$ . Namely, two different runs of the same element in  $\hat{x}$  correspond to elements in  $P$  with the same down-sets but with different up-sets — this is a fact that is not explicitly mentioned in [1, Section 4] but it can be proved.  $\square$

We have the following immediate corollary to Proposition 4.

**Corollary 5.** *Primitive  $(2 + 2)$ -free posets on  $n$  elements are in one-to-one correspondence with primitive ascent sequences of length  $n$ .*

One more corollary follows from the proof of Proposition 4.

**Corollary 6.** *The statistic  $\text{rep}$  on  $\mathcal{P}_n^{(k)}$  corresponds to the statistic  $\text{epairs}$  on  $\text{Asc}_n^{(k)}$  under  $\mathfrak{B}$ .*

**2.3. Restricted matrices and ascent sequences.** In Dukes and Parviainen [3] a bijection  $\Gamma : \mathcal{M}_n \rightarrow \text{Asc}_n$  was presented. Here we find it convenient to describe the inverse  $\zeta : \text{Asc}_n \rightarrow \mathcal{M}_n$  of this map. Given  $A \in \mathcal{M}_n$ , let  $\text{index}(A)$  be the smallest value of  $i$  such that  $A_{i, \dim(A)}$  is nonzero. Given a value  $m$  such that  $0 \leq m \leq \dim(A)$ , we define the matrix  $\phi(A, m)$  according to the following:

- (i) If  $0 \leq m < \text{index}(A)$  then let  $\phi(A, m)$  be the matrix  $A$  with the entry at position  $(m + 1, \dim(A))$  increased by 1.
- (ii) If  $\text{index}(A) \leq m \leq \dim(A)$  then we form  $\phi(A, m)$  in the following way. Let  $A'$  be the matrix with  $\dim(A') = \dim(A) + 1$  formed by inserting a row of zeros immediately after row  $m$  of  $A$ , and a column of zeros immediately after column  $m$  of  $A$ . Let  $A'_{m+1, \dim(A)+1} = 1$ . Swap the values  $A'_{i, m+1}$  and  $A'_{i, \dim(A)+1}$  for all  $1 \leq i \leq m$ . Call the resulting matrix  $\phi(A, m)$ .

As an example of the second construction, let

$$A = \begin{pmatrix} 1 & 7 & 1 \\ 0 & 9 & 3 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then  $\phi(A, 2)$  is given by

$$\begin{pmatrix} 1 & 7 & 1 \\ 0 & 9 & 3 \\ 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 7 & 0 & 1 \\ 0 & 9 & 0 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 7 & 1 & 0 \\ 0 & 9 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \phi(A, 2).$$

Given  $x = (x_1, \dots, x_n) \in \text{Asc}_n$ , let  $\epsilon$  be the empty matrix. Define  $\phi(\epsilon, 0) := (1)$  and

$$\zeta(x) = \phi(\dots \phi(\phi(\epsilon, x_1), x_2) \dots, x_n).$$

**Proposition 7.** *For each  $n \geq 0$  and  $k \geq 1$ , we have  $\zeta(x) \in \mathcal{M}_n^{(k)}$  if and only if  $x \in \text{Asc}_n^{(k)}$ .*

*Proof.* Let  $x = (x_1, \dots, x_n) \in \text{Asc}_n$ . Define  $A^{(i)} = \zeta(x_1, \dots, x_i)$  for all  $1 \leq i \leq n$ . Let us suppose that  $x \notin \text{Asc}_n^{(k)}$  so that there exists  $i$  such that  $x_i = x_{i+1} = \dots = x_{i+k} = c$ . Since  $A^{(i)} = \phi(A^{(i-1)}, x_i)$ , we have  $\text{index}(A^{(i)}) - 1 = x_i = c$ . Let  $d = \dim(A^{(i)})$ . So the entry  $A_{c+1, d}^{(i)} \geq 1$ . Since  $A^{(i+1)} = \phi(A^{(i)}, c)$ , and  $x_{i+1} = c < \text{index}(A^{(i)}) = 1 + c$ , the rule (i) is used and we have  $A^{(i+1)}$  as the matrix  $A^{(i)}$  with the entry at position  $(c + 1, d)$  increased by 1. So  $A_{c+1, d}^{(i+1)} \geq 2$ . Doing this repeatedly, we find that  $A_{c+1, d}^{(i+k)} \geq 1 + k$ , which means that  $A^{(i+k)} \notin \text{Asc}_{i+k}^{(k)}$ , and so  $A^{(n)} \notin \mathcal{M}_n^{(k)}$  since neither of the construction rules (i) or (ii) can decrease an entry of a matrix (although entries may be moved).



Next we prove that  $\zeta(x) \notin \mathbf{M}_n^{(k)} \Rightarrow x \notin \text{Asc}_n^{(k)}$ . The inverse of  $\zeta$  was recursively described in [3]. In order to find the ascent sequence  $(x_1, \dots, x_n)$  corresponding to  $A \in \mathbf{M}_n$ , one finds that there is a unique  $f(A) \in \mathbf{M}_{n-1}$  and value  $x_n = \text{index}(A) - 1$  such that  $A = \phi(f(A), x_n)$  and  $f(A) = \zeta(x_1, \dots, x_{n-1})$ . To determine the reduced matrix  $f(A)$  one must invoke one of the three removal rules, called **Rem1** – **Rem3** in [3]. We present the argument without describing these rules explicitly.

Let  $X = \zeta(x) \in \mathbf{M}_n \setminus \mathbf{M}_n^{(k)}$ . Then there is at least one entry  $X_{ab}$  in  $X$  with  $X_{ab} \geq k + 1$ . At some stage during the deconstruction process, the value  $X_{ab}$  will be in the rightmost column of  $f(f(\dots f(A) \dots))$ . If there are non negative values above it, they will be removed in due course of the deconstruction. One then has a matrix  $B \in \mathbf{M}_m$ , where  $\text{value}(B) = X_{ab}$  and

$$A = \phi(\dots \phi(\phi(B, x_{m+1}), x_{m+2}) \dots, x_n).$$

Since  $X_{ab} \geq k + 1$ , the next  $k$  removals will invoke **Rem1**, thereby giving  $x_{m-1} = x_{m-2} = \dots = x_{m-k}$ . Since  $\text{value}(B) \geq 1$ , regardless of which removal rule is used next, one finds that  $x_{m-k-1} = x_{m-k}$ . This implies there are at least  $k + 1$  consecutive entries in the ascent sequence which take the same value. Hence  $x \notin \text{Asc}_n^{(k)}$ .  $\square$

### 3. ENUMERATING ASCENT SEQUENCES WITH RESTRICTED RUNS

The primitive ascent sequences of length  $n$  are in one-to-one correspondence with matrices in  $\mathbf{M}_n^{(1)}$ , see [3, Thm. 13]. Jovovic [8] conjectured the generating function (1) for the number of matrices in  $\mathbf{M}_n^{(1)}$  (see [12, A138265]). Here we prove this conjecture (Theorem 8) by using the bijective correspondence with ascent sequences, and we also generalize the generating function (1) to count more complicated objects (Theorem 9).

In Bousquet-Mélou et al. [1] it was shown that

$$P(x) = \sum_{a \in \text{Asc}} x^{|a|} = \sum_{n \geq 0} \prod_{i=1}^n \left(1 - (1-x)^i\right). \quad (6)$$

Let  $K(x) = \sum_{n \geq 0} k_n x^n$  where  $k_n = |\text{Asc}_n^{(1)}|$  is the number of primitive ascent sequences of length  $n$ . Due to Propositions 4 and 7 we have

$$K(x) = \sum_{n \geq 0} |\mathbf{M}_n^{(1)}| x^n = \sum_{n \geq 0} |\mathbf{P}_n^{(1)}| x^n.$$

We now give an explicit formula for  $K(x)$ , proving a conjecture of Jovovic [8].

**Theorem 8.** *We have*

$$K(x) = \sum_{n \geq 0} \prod_{i=0}^n \left(1 - \frac{1}{(1+x)^i}\right).$$

*Proof.* Every ascent sequence  $a = (a_1, \dots, a_n)$  may be written uniquely in the form

$$(b_1^{m_1}, \dots, b_k^{m_k})$$

where  $(b_1, \dots, b_k)$  is a primitive ascent sequence, and  $m_i$  is the number of consecutive entries of  $b_i$  in  $a$ . For example, if  $a = (0, 0, 1, 1, 1, 0, 2, 2, 3, 1, 1, 0, 4)$  then  $a = (0^2, 1^3, 0^1, 2^2, 3^1, 1^2, 0^1, 4^1)$  and  $b = (0, 1, 0, 2, 3, 1, 0, 4)$  is the underlying primitive ascent sequence with multiplicities  $(2, 3, 1, 2, 1, 2, 1, 1)$ . A primitive ascent sequence of length  $n \geq 1$  gives rise to an infinite number of ascent sequences by

choosing multiplicities  $(m_1, \dots, m_n) \in \mathbb{N}^n$ . Therefore,

$$P(t) = \sum_{n \geq 0} k_n (t + t^2 + \dots)^n = \sum_{n \geq 0} k_n \left( \frac{t}{1-t} \right)^n = K \left( \frac{t}{1-t} \right). \quad (7)$$

Setting  $x = t/(1-t)$ , we see that  $t = x/(1+x)$  so that

$$K(x) = P \left( \frac{x}{1+x} \right) = \sum_{n \geq 0} \prod_{i=1}^n \left( 1 - \frac{1}{(1+x)^i} \right). \quad (8)$$

□

Let

$$B_k(x) = \sum_{n \geq 0} |\text{Asc}_n^{(k)}| x^n = \sum_{n \geq 0} |\text{M}_n^{(k)}| x^n = \sum_{n \geq 0} |\text{P}_n^{(k)}| x^n,$$

where the latter two identities were established in Theorem 7 and Proposition 4. Then we have the following theorem which generalizes Theorem 8 (the case  $k = 1$ ) and gives the generating function for the number of ascent sequences that have a run of length at most  $k$ .

**Theorem 9.** *We have*

$$\sum_{n \geq 0} |\text{Asc}_n^{(k)}| x^n = \sum_{n \geq 0} |\text{M}_n^{(k)}| x^n = \sum_{n \geq 0} |\text{P}_n^{(k)}| x^n = \sum_{n \geq 0} \prod_{i=1}^n \left( 1 - \left( \frac{1-x}{1-x^{k+1}} \right)^i \right).$$

*Proof.* It is easy to see that

$$B_k(x) = \sum_{n \geq 0} k_n (x + x^2 + \dots + x^k)^n = K \left( \frac{x(x^k - 1)}{(x - 1)} \right) = \sum_{n \geq 0} \prod_{i=1}^n \left( 1 - \left( \frac{1-x}{1-x^{k+1}} \right)^i \right).$$

□

#### 4. ENUMERATION OF ASCENT SEQUENCES BY ASCENTS, EQUAL PAIRS, AND LAST LETTER

The theorems in this section concern the enumeration of ascent sequences. Let

$$G(u, v, y, t) = \sum_{s \in \text{Asc}} u^{\text{asc}(s)} v^{\text{last}(s)} y^{\text{epairs}(s)} t^{|s|} = \sum_{a, m, b, n \geq 0} G_{a, m, b, n} u^a v^m y^b t^n \quad (9)$$

be the generating function for ascent sequences according to the statistics introduced in Section 1. The value  $G_{a, b, m, n}$  is the number of ascent sequences of length  $n$  with  $a$  ascents,  $b$  equal pairs, and last letter  $m$ .

From the correspondences in [1, 3] and Corollary 6, we see that this generating function is also the generating function of  $(\mathbf{2} + \mathbf{2})$ -free posets and our upper-triangular matrices:

$$G(u, v, y, t) = \sum_{P \in \mathcal{P}} u^{\text{levels}(P)} v^{\text{minmax}(P)} y^{\text{rep}(P)} t^{|P|} \quad (10)$$

$$= \sum_{A \in \mathcal{M}} u^{\dim(A)-1} v^{\text{index}(A)-1} y^{\text{extra}(A)} t^{|A|}. \quad (11)$$

Let  $H(u, v, y, t) = G(u, v, y, t) - 1$  be the generating function for these statistics over all nonempty ascent sequences.

**Lemma 10.** *The formal power series  $H(u, v, y, t)$  satisfies*

$$\begin{aligned} & H(u, v, y, t)(v - 1 - t - tyv + ty + tuv) \\ & = t(v - 1) - tH(u, 1, y, t) + tuv^2 H(uv, 1, y, t). \end{aligned} \quad (12)$$

*Proof.* It is easy to see that

$$\begin{aligned}
 G(u, v, y, t) &= 1 + t + t \sum_{\substack{n \geq 1, \\ a, b, m \geq 0}} G_{a, b, m, n} t^n \left( \left( \sum_{i=0}^{m-1} u^a v^i y^b \right) + u^a v^m y^{b+1} + \sum_{i=m+1}^{a+1} u^{a+1} v^i y^b \right) \\
 &= 1 + t + t \sum_{\substack{n \geq 1, \\ a, b, m \geq 0}} G_{a, b, m, n} t^n u^a y^b \left( \frac{v^m - 1}{v - 1} + y v^m + u \frac{v^{a+2} - v^{m+1}}{v - 1} \right) \\
 &= 1 + t + t(G(u, v, y, t) - 1) \left( \frac{1 + y(v - 1) - uv}{v - 1} \right) - \frac{t}{v - 1} (G(u, 1, y, t) - 1) \\
 &\quad + \frac{tuv^2}{v - 1} (G(uv, 1, y, t) - 1).
 \end{aligned}$$

Since  $G(u, v, y, t) = 1 + H(u, v, y, t)$  we find that

$$\begin{aligned}
 (v - 1)H(u, v, y, t) &= t(v - 1) + H(u, v, y, t)(t + tyv - ty - tuv) \\
 &\quad - tH(u, 1, y, t) + tuv^2 H(uv, 1, y, t).
 \end{aligned}$$

□

We use the above lemma to give an expression for the power series  $G(u, 1, y, t)$ .

**Theorem 11.** *We have*

$$G(u, 1, y, t) = 1 + \frac{t(1 - u)}{\Delta_1} + \sum_{n=1}^{\infty} \frac{t(1 - u)(1 - ty)^n (1 + t - ty)^n \prod_{i=1}^n \Gamma_i}{\Delta_n \Delta_{n+1}},$$

where  $\Delta_k = (1 - ty)^k (1 - u) + u(1 + t - ty)^k$  and  $\Gamma_k = (u(1 + t - ty)^k) / \Delta_k$ .

*Proof.* The left hand side of the functional equation (12) vanishes when the coefficient to  $H(u, v, y, t)$  is zero. This happens precisely when  $v$  is

$$W(u, y, t) = \frac{1 + t - ty}{1 + tu - ty}. \quad (13)$$

Replacing  $v$  by  $W(u, y, t)$  in (12) gives

$$0 = \frac{t^2(1 - u)}{1 + tu - ty} - tH(u, 1, y, t) + tu \left( \frac{1 + t - ty}{1 + tu - ty} \right)^2 H \left( u \frac{1 + t - ty}{1 + tu - ty}, 1, y, t \right)$$

and hence

$$H(u, 1, y, t) = \frac{t(1 - u)}{1 + tu - ty} + u \left( \frac{1 + t - ty}{1 + tu - ty} \right)^2 H \left( u \frac{1 + t - ty}{1 + tu - ty}, 1, y, t \right). \quad (14)$$

Next let

$$\Delta_k = (1 - ty)^k (1 - u) + u(1 + t - ty)^k.$$

It is easy to check that  $\Delta_1 = 1 + tu - ty$ . Also let

$$\Gamma_k = \frac{u(1 + t - ty)^k}{\Delta_k}.$$

The following identities are immediate:

$$\begin{aligned}
 (1 - u)|_{u=\Gamma_s} &= \frac{\Delta_s}{\Delta_s} - \frac{u(1 + t - ty)^s}{\Delta_s} = \frac{(1 - ty)^s (1 - u)}{\Delta_s}, \\
 \Delta_k|_{u=\Gamma_s} &= \frac{(1 - ty)^k (1 - ty)^s (1 - u)}{\Delta_s} + \frac{u(1 + t - ty)^s (1 + t - ty)^k}{\Delta_s} = \frac{\Delta_{k+s}}{\Delta_s}, \\
 \frac{(1 - u)}{\Delta_k}|_{u=\Gamma_s} &= \frac{\Delta_s}{\Delta_{k+s}} \frac{(1 - ty)^s (1 - u)}{\Delta_s} = \frac{(1 - ty)^s (1 - u)}{\Delta_{k+s}},
 \end{aligned}$$

$$\Gamma_k|_{u=\Gamma_s} = (1+t-ty)^k \frac{u(1+t-ty)^s}{\Delta_s} \frac{\Delta_s}{\Delta_{s+k}} = \Gamma_{s+k}.$$

We can then rewrite (14) as

$$H(u, 1, y, t) = \frac{t(1-u)}{\Delta_1} + \frac{(1+t-ty)}{\Delta_1} \Gamma_1 H(\Gamma_1, 1, y, t). \quad (15)$$

Iterating (15) gives

$$\begin{aligned} H(u, 1, y, t) &= \frac{t(1-u)}{\Delta_1} + \frac{(1+t-ty)}{\Delta_1} \Gamma_1 \left\{ t \frac{(1-ty)(1-u)}{\Delta_1} \frac{\Delta_1}{\Delta_2} \right. \\ &\quad \left. + (1+t-ty) \frac{\Delta_1}{\Delta_2} \Gamma_2 G(\Gamma_2, 1, y, t) \right\} \\ &= \frac{t(1-u)}{\Delta_1} + \frac{t(1-u)(1-ty)(1+t-ty)\Gamma_1}{\Delta_1 \Delta_2} + \\ &\quad \frac{(1+t-ty)^2 \Gamma_1 \Gamma_2}{\Delta_2} H(\Gamma_2, 1, y, t). \end{aligned} \quad (16)$$

If we iterate (16), then we find that

$$\begin{aligned} H(u, 1, y, t) &= \frac{t(1-u)}{\Delta_1} + \frac{t(1-u)(1-ty)(1+t-ty)\Gamma_1}{\Delta_1 \Delta_2} \\ &\quad + \frac{t(1-u)(1-ty)^2(1+t-ty)^2 \Gamma_1 \Gamma_2}{\Delta_2 \Delta_3} \\ &\quad + \frac{t(1-u)(1-ty)^3(1+t-ty)^3 \Gamma_1 \Gamma_2 \Gamma_3}{\Delta_3 \Delta_4} \\ &\quad + \frac{(1+t-ty)^4 \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4}{\Delta_4} H(\Gamma_4, 1, y, t). \end{aligned} \quad (17)$$

One can then easily prove by induction that

$$\begin{aligned} H(u, 1, y, t) &= \frac{t(1-u)}{\Delta_1} + \sum_{n=1}^{2^n-1} \frac{t(1-u)(1-ty)^n(1+t-ty)^n \prod_{i=1}^n \Gamma_i}{\Delta_n \Delta_{n+1}} + \\ &\quad \frac{(1+t-ty)^{2^n} \prod_{i=1}^{2^n} \Gamma_i}{\Delta_{2^n}} H(\Gamma_{2^n}, 1, y, t). \end{aligned} \quad (18)$$

Since each  $\Gamma_i$  has a factor of  $u$ , it is easy to see that, as a formal power series in  $u$ ,

$$H(u, 1, y, t) = \frac{t(1-u)}{\Delta_1} + \sum_{n=1}^{\infty} \frac{t(1-u)(1-ty)^n(1+t-ty)^n \prod_{i=1}^n \Gamma_i}{\Delta_n \Delta_{n+1}}. \quad (19)$$

□

The first few terms of  $G(u, 1, y, t)$  are

$$G(u, 1, y, t) = 1 + \frac{P_0(t, y)}{(1-ty)} + \frac{P_1(t, y)}{(1-ty)^3} u + \frac{P_2(t, y)}{(1-ty)^6} u^2 + \frac{P_3(t, y)}{(1-ty)^{10}} u^3 + O(u^4) \quad (20)$$

where the power series  $P_i(t, y)$  are given in Figure 1. For example, for the ascent sequences with a single ascent one can see that

$$\begin{aligned} \frac{P_1(t, y)}{(1-ty)^3} &= \frac{t^2(1-ty) + t^3}{(1-ty)^3} = \frac{t^2}{(1-ty)^2} + \frac{t^3}{(1-ty)^3} \\ &= \sum_{n \geq 2} (n-1) y^{n-2} t^n + \sum_{n \geq 3} \binom{n-1}{2} y^{n-3} t^n. \end{aligned}$$

Here the first sum accounts for ascent sequences of the form  $0^a 1^b$  where  $a, b \geq 1$  and the second sum accounts for ascent sequences of the form  $0^a 1^b 0^c$  where  $a, b, c \geq 1$ .

$i$	$P_i(t, y)$
0	$t$
1	$t^2(1 - ty) + t^3$
2	$t^3 + 4t^4 + 4t^5 + t^6 - 3t^4y - 8t^5y - 4t^6y + 3t^5y^2 + 4t^6y^2 - t^6y^3$
3	$t^4 + 11t^5 + 33t^6 + 42t^7 + 26t^8 + 8t^9 + t^{10} - 6t^5y - 55t^6y - 132t^7y - 126t^8y - 52t^9y - 8t^{10}y + 15t^6y^2 + 110t^7y^2 + 198t^8y^2 + 126t^9y^2 + 26t^{10}y^2 - 20t^7y^3 - 110t^8y^3 - 132t^9y^3 - 42t^{10}y^3 + 15t^8y^4 + 55t^9y^4 + 33t^{10}y^4 - 6t^9y^5 - 11t^{10}y^5 + t^{10}y^6$

FIGURE 1. The first four power series  $P_i(t, y)$ .

We can now use Lemma 10 and Theorem 11 to give an expression for  $G(u, v, y, t)$ . That is, if we define  $\Delta_0 = 1$ , then by Theorem 11, we have that

$$G(u, 1, y, t) = 1 + \sum_{n \geq 0} \frac{t(1-u)(1-ty)^n(1+t-ty)^n \prod_{i=1}^n \Gamma_i}{\Delta_n \Delta_{n+1}} \quad (21)$$

and

$$G(uv, 1, y, t) = 1 + \sum_{n \geq 0} \frac{t(1-uv)(1-ty)^n(1+t-ty)^n \prod_{i=1}^n \bar{\Gamma}_i}{\bar{\Delta}_n \bar{\Delta}_{n+1}} \quad (22)$$

where  $\bar{\Delta}_0 = 1$  and  $\bar{\Delta}_k = (1-ty)^k(1-uv) + uv(1+t-ty)^k$  and  $\bar{\Gamma}_k = (uv(1+t-ty)^k)/\bar{\Delta}_k$  for  $k \geq 1$ . Thus we have the following theorem.

**Theorem 12.**

$$\begin{aligned} G(u, v, y, t) &= 1 + \frac{t}{(v-1-t-tyv+ty+tuv)} \left( v-1 \right. \\ &\quad \left. - t \sum_{n \geq 0} (1-ty)^n(1+t-ty)^n \left\{ \frac{(1-u) \prod_{i=1}^n \Gamma_i}{\Delta_n \Delta_{n+1}} - \frac{uv^2(1-uv) \prod_{i=1}^n \bar{\Gamma}_i}{\bar{\Delta}_n \bar{\Delta}_{n+1}} \right\} \right). \end{aligned}$$

The first few terms of this power series are as follows:

$$\begin{aligned} G(u, v, y, t) &= 1 + t + (uv + y)t^2 + (u + u^2v^2 + 2uvy + y^2)t^3 \\ &\quad + (u^2 + 2u^2v + u^2v^2 + u^3v^3 + 3uy + 3u^2v^2y + 3uvy^2 + y^3)t^4 \\ &\quad + O(t^5). \end{aligned}$$

**4.1. Enumeration of primitive ascent sequences by ascents.** Primitive ascent sequences, that is, ascent sequences with no 2-runs, correspond to setting  $y = 0$  in  $G(u, 1, y, t)$ . When  $y = 0$ , the expression  $\Delta_k$  becomes  $(1-u) + u(1+t)^k$  and  $\Gamma_k$  becomes  $u(1+t)^k/\delta_k$ . Thus we have the following;

**Corollary 13.** Let  $\delta_k = (1-u) + u(1+t)^k$  and  $\gamma_k = u(1+t)^k/\delta_k$ . Then

$$G(u, 1, 0, t) = 1 + \frac{t(1-u)}{\delta_1} + \sum_{n=1}^{\infty} \frac{t(1-u)(1+t)^n \prod_{i=1}^n \gamma_i}{\delta_n \delta_{n+1}}. \quad (23)$$

Unfortunately we cannot derive a generating function for the number of primitive ascent sequences from  $G(u, 1, 0, t)$  by setting  $u = 1$  (this generating function is derived in Section 3 using different arguments). The power series for the first few terms in the expansion of  $G(u, 1, 0, t)$  (about  $u = 0$ ),

$$G(u, 1, 0, t) = 1 + \sum_{n=0}^4 q_n(t)u^n + O(u^5) \quad (24)$$

$i$	$q_i(t)$
0	$t$
1	$t^2 + t^3$
2	$t^3 + 4t^4 + 4t^5 + t^6$
3	$t^4 + 11t^5 + 33t^6 + 42t^7 + 26t^8 + 8t^9 + t^{10}$
4	$t^5 + 26t^6 + 171t^7 + 507t^8 + 840t^9 + 865t^{10} + 584t^{11} + 262t^{12} + 76t^{13} + 13t^{14} + t^{15}$
5	$t^6 + 57t^7 + 718t^8 + 4017t^9 + 12866t^{10} + 26831t^{11} + 39268t^{12} + 42211t^{13} + 34221t^{14} + 21184t^{15} + 10015t^{16} + 3571t^{17} + 933t^{18} + 169t^{19} + 19t^{20} + t^{21}$

FIGURE 2. The first six power series  $q_i(t)$ .

are given in Figure 2. Note that the power series  $q_n(t)$  are unimodal for  $0 \leq n \leq 7$ . It would be nice to have a combinatorial proof of this for general  $n$ .

#### 5. PERMUTATIONS AND MATCHINGS CORRESPONDING TO ASCENT SEQUENCES WITH BOUNDED RUN-LENGTH

We conclude by mentioning the restricted permutations and matchings that correspond, via the maps in [1, 3], to ascent sequences with bounded run-length. First, we recall a few definitions from the papers [1, 2].

Let  $V = \{v_1, v_2, \dots, v_n\}$  with  $v_1 < v_2 < \dots < v_n$  be any finite subset of  $\mathbb{N}$ . The *standardization* of a permutation  $\pi$  of the elements of  $V$  is the permutation  $\text{std}(\pi)$  of  $\{1, \dots, n\}$  obtained from  $\pi$  by replacing the letter  $v_i$  with the letter  $i$ . As an example,  $\text{std}(39685) = 15342$ . Let

$$\mathcal{R}_n = \{\pi_1 \dots \pi_n \in \mathcal{S}_n : \text{if } \text{std}(\pi_i \pi_j \pi_k) = 231 \text{ then } j \neq i + 1 \text{ or } \pi_i \neq \pi_k + 1\},$$

where  $\mathcal{S}_n$  is the set of permutations of  $\{1, 2, \dots, n\}$ .

In other words,  $\mathcal{R}_n$  is the set of permutations of  $[n]$  where, in each occurrence of the pattern 231, either the letters corresponding to the 2 and the 3 are nonadjacent, or else the letters corresponding to the 2 and the 1 are not adjacent in value. For instance, the occurrence 463 in  $\pi = 546123$  violates both conditions, since 4 and 6 are adjacent letters in  $\pi$  and 4 and 3 are adjacent values. Note that both  $\mathcal{R}_n$  and  $\mathcal{T}_n$  are defined in terms of avoidance of *bivincular patterns*, which were defined in [1].

Also, let  $\mathcal{T}_n$  be the subset of  $\mathcal{R}_n$  whose permutations have no adjacent letters that are adjacent in value and in decreasing order, that is, no descent consisting of letters that differ in size by one. In the permutation 546123, mentioned above, there is one violation of that condition, namely the 54.

Let  $\mathcal{R}_n^{(k)}$  be the subset of permutations  $\pi \in \mathcal{R}_n$  such that there do not exist integers  $i$  and  $m$  with  $\pi_i = m$ ,  $\pi_{i+1} = m - 1$ ,  $\dots$ ,  $\pi_{i+k} = m - k$ . Also, for any general pattern  $p$ , let  $p(\pi)$  be the number of occurrences of  $p$  in  $\pi$ .

A *matching* of the set  $[2n] = \{1, 2, \dots, 2n\}$  is a partition of  $[2n]$  into subsets of size 2, each of which is called an *arc*. The smaller number in an arc is its *opener* and the larger one its *closer*. A matching is said to be *Stoimenow* if it has no pair of arcs  $\{a, b\}$  and  $\{c, d\}$ , with  $a < b$  and  $c < d$ , satisfying one (or both) of the following conditions:

- (1)  $a = c + 1$  and  $b < d$ ,
- (2)  $a < c$  and  $b = d + 1$ .

In other words, a Stoimenow matching has no pair of arcs such that one is nested within the other and the openers, or closers, of the two arcs differ by 1.

Let  $\text{Match}_n$  denote the set of Stoimenow matchings on  $[2n]$  and  $\text{Match}$  the set of all such matchings. If  $(i, j)$  and  $(i+1, j+1)$  are arcs in a matching  $M$ , we say that they are *similar*. Let  $\text{echords}(M)$  be the minimum number of arcs in  $M$  one has to remove to obtain a matching without similar arcs. Let  $\text{Match}_n^{(k)}$  be the collection of matchings  $M \in \text{Match}_n$  such that for no pair  $i$  and  $j$  do all of  $(i, j), (i+1, j+1), \dots, (i+k, j+k)$  belong to  $M$ .

Bijections  $\Lambda : \mathcal{R}_n \rightarrow \text{Asc}_n$  and  $\Psi' : \text{Match}_n \rightarrow \text{Asc}_n$  were presented in [1, Thm. 1] and [2, Thm. 7], respectively. Let us write  $\Upsilon$  and  $\Omega$  for their respective inverses, so that we have  $\Upsilon : \text{Asc}_n \rightarrow \mathcal{R}_n$  and  $\Omega : \text{Asc}_n \rightarrow \text{Match}_n$ . It is then fairly easy to prove the following theorem and corollary, and we omit these proofs.

**Theorem 14.** *We have*

- (i)  $\Upsilon(\text{Asc}_n^{(k)}) = \mathcal{R}_n^{(k)}$ , and
- (ii)  $\Omega(\text{Asc}_n^{(k)}) = \text{Match}_n^{(k)}$ .

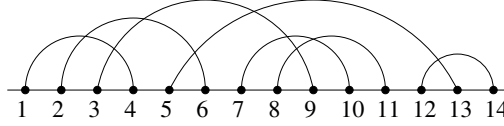
*In particular,*

- (iii)  $\Upsilon(\text{Asc}_n^{(1)}) = \mathcal{T}_n$ , and
- (iv)  $\Omega(\text{Asc}_n^{(1)}) = \{M \in \text{Match}_n : \text{echords}(M) = 0\}$ .

For a permutation  $\pi$ , let  $\text{adjdes}(\pi)$  be the number of descents in  $\pi$  whose letters differ by one in size. For instance,  $\text{adjdes}(2543176) = 3$ , accounted for by 54, 43 and 76.

**Corollary 15.** *Given  $x \in \text{Asc}$ , we have  $\text{epairs}(x) = \text{echords}(\Omega(x)) = \text{adjdes}(\Upsilon(x))$ .*

**Example 16.** Given the ascent sequence  $x = (0, 1, 1, 0, 2, 0, 1)$  the corresponding permutation is  $\Upsilon(x) = 6417325$  and the corresponding matching is  $\Omega(x)$ :



Note that  $\text{adjdes}(6417325) = 1$  because we have the two adjacent entries  $\pi_5\pi_6 = 32$ . Also  $\text{echords}(\Omega(x)) = 1$  since we have one pair of similar arcs in  $\Omega(x)$ , namely  $(7, 10), (8, 11)$ .

For a matching  $M \in \text{Match}_n$  let  $|M| = n$  and, for  $n \geq 1$ , let  $A^*$  denote the arc in  $M$  having the rightmost closer. Let  $\text{cruns}(M)$  be the number of runs of closers to the left of  $A^*$ . Moreover, let  $\text{larcs}(M)$  be the number of runs of closers to the left of the arc having the closer next to the right of  $A^*$ 's opener. For the matching  $M$  in Example 16,  $|M| = 7$ ,  $A^* = (12, 14)$ ,  $\text{cruns}(M) = 3$  (the runs of closers are 4, 6, 9(10)(11)), and  $\text{larcs}(M) = 1$  (there is one run of closers, 4, to the left of  $(5, 13)$ ).

Given  $\pi \in \mathcal{R}_n$ , let us label the positions of  $\pi$  from left to right where we can insert  $(n+1)$  in order to create  $\pi' \in \mathcal{R}_{n+1}$ . Define  $b(\pi)$  to be the label immediately to the left of  $n$  in  $\pi$ . For example, if  $\pi = 6132547 \in \mathcal{R}_7$ , then  $\pi$  is labeled as  $061_132_254_37_4$  and  $b(\pi) = 3$  since 3 is the label immediately to the left of 7.

Let  $\mathcal{R} = \bigcup_{n \geq 0} \mathcal{R}_n$ . Using the properties of the corresponding bijections in [1] and [2],

$$\begin{aligned} G(u, v, y, t) &= \sum_{\pi \in \mathcal{R}} u^{\text{asc}(\pi^{-1})} v^{b(\pi)} y^{\text{adjdes}(\pi)} t^{|\pi|} \\ &= \sum_{M \in \text{Match}} u^{\text{cruns}(M)} v^{\text{larcs}(M)} y^{\text{echords}(M)} t^{|M|}. \end{aligned}$$

Thus, Theorem 12 provides the generating function for the number of permutations and matchings in question subject to 3 statistics.

Finally, as a corollary to Theorems 9 and 14, we have the following enumerative result.

**Theorem 17.** 
$$\sum_{n \geq 0} |\mathcal{R}_n^{(k)}| x^n = \sum_{n \geq 0} |\text{Match}_n^{(k)}| x^n = \sum_{n \geq 0} \prod_{i=1}^n \left( 1 - \left( \frac{1-x}{1-x^k} \right)^i \right).$$

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